Int. J. Solids Structures Vol. 30, No. 20, pp. 2757–2778, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00
Pergamon Press Ltd

DAMAGE OF FIBER-REINFORCED COMPOSITE MATERIALS WITH MICROMECHANICAL CHARACTERIZATION

GEORGE Z. VOYIADJIS and PETER I. KATTAN
Department of Civil Engineering, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

(*Received* 10 *November* 1992; *in revised form* 19 *March* 1993)

Abstract—A damage mechanics model is presented to characterize brittle failure in elastic fiber-reinforced composite materials. An overall fourth-rank damage effect tensor is introduced based on the hypothesis of elastic energy equivalence to account for the overall damage of the composite system. In addition, two local (matrix and fiber) fourth-rank damage effect tensors are introduced to account for the local effects of damage experienced by both the matrix and the fibers. The overall and local damage tensors are correlated together using micromechanical considerations.

The explicit constitutive equations for the damaged material are derived for a uniaxially loaded unidirectional thin fiber-reinforced composite lamina. The model is also applied to a unidirectional thin lamina under a state of plane stress. New expressions are derived for the stress and strain concentration factors for the damaged material in terms of the undamaged concentration factors and the damage variables. In addition, explicit expressions are obtained for the overall and local fourth-rank damage effect tensors. The research presented in this work is the three-dimensional generalization of the uniaxial tension model formulated previously by the authors.

1. INTRODUCTION

Fiber-reinforced composite materials play an important role in the industry today through the design and manufacture of advanced materials capable of attaining higher stiffness/density and strength/density ratios. Of particular importance is the problem of damage initiation and evolution in fiber-reinforced metal matrix composite plates. Although the literature is rich in new developments in the composite materials technology, it lacks tremendously a consistent analysis of damage mechanisms in composite materials.

In the analysis of composite materials, one can follow a continuum approach or a micromechanical approach. In the continuum approach, the composite material is treated as an orthotropic or transversely isotropic medium. Then the classical equations of orthotropic elasticity are used in the analysis (Talreja, 1985, 1986; Christensen, 1988, 1990). No distinction is made between the matrix and the fibers in this approach and therefore, this approach lacks the ability to account for local effects and especially the effects of the matrix–fiber interaction. There were some attempts to include damage using the continuum approach (Talreja, 1985; Shen *et al.*, 1985; Lene, 1986). However, these attempts lack the distinction between matrix and fiber damage or damage due to the matrix–fiber interaction.

During the past two decades, researchers have been using micromechanical methods in the analysis of composite materials. The advantages of using such methods are that local effects can be accounted for and different damage mechanisms can be identified. Hill (1965, 1972) introduced volume averages of stress and strain increments in the matrix and fibers and introduced certain concentration factors to relate these volume averages to the overall uniform increments. Dvorak and Bahei-El-Din (1979, 1982, 1987) used Hill's method to analyse the elasto-plastic behavior of fiber-reinforced composite materials. They considered elastic fibers embedded in an elasto-plastic matrix and identified two distinct deformation modes; matrix dominated and fiber dominated. They concluded that the fiber-dominated mode is general in the sense that it can be treated as a general case of plastic deformation of a heterogeneous medium.

A thermomechanical constitutive theory has been recently proposed by Allen and Harris (1987) to analyse distributed damage in elastic composites. In particular, the problem of matrix cracking has been extensively studied in the literature (Dvorak *et al.*, 1985; Dvorak and Laws, 1987; Laws and Dvorak, 1987; Allen *et al.*, 1987, 1988; Lee *et al.*, 1989).

The theoretical formulation presented here is based on the concept of effective stress that was originally proposed by Kachanov (1958). The pioneering work of Kachanov (1958) started what is now known as continuum damage mechanics. Different researchers (Lemaitre, 1985, 1986; Chaboche, 1988a, b; Krajcinovic, 1983, 1984) used continuum damage mechanics to analyse different types of damage in materials ranging from brittle fracture to ductile failure. However, no attempt has been made to use the concepts of continuum damage mechanics to analyse damage in composite materials using the micromechanical approach. It should be mentioned that some researchers (Talreja, 1985) have used it to analyse damage in composite materials using the continuum approach by modeling the composite as a transversely isotropic medium.

In this work, continuum damage mechanics is used with a micromechanical composite model to analyse damage in composite materials. Both overall and local damage variables are introduced to model the overall and local damage effects. Stress and strain concentration factors are derived for the damaged composite. The model is applied in detail to a unidirectional thin lamina that is subjected to uniaxial tension. It is also applied to a unidirectional thin lamina under a state of plane stress. The research presented in this work is the three-dimensional generalization of the uniaxial tension model derived previously by the authors (Kattan and Voyiadjis, 1993).

2. THEORETICAL FORMULATION

2.1. *Definitions and assumptions*

Consider a body of fiber-reinforced composite material in the initial undeformed and undamaged configuration C_0. Let C be the configuration of the body that is both damaged and deformed after a set of external agencies act on it. Following the concept of effective stress (Kachanov, 1958; Murakami, 1988), consider a fictitious configuration of the body $\bar{C}$ obtained from C by removing all the damage that the body (both matrix and fibers) has undergone, i.e. $\bar{C}$ is the state of the body after it had deformed without damage. Assume that the representative volume element in C_0 is statistically homogeneous, and is free of voids and cracks initially. Assume also that the composite is loaded by an overall stress or strain field which is followed by increments of loading. The overall stress or strain fields are assumed to be uniform. The effective overall stress is defined in the configuration $\bar{C}$ as the stress in a perfectly-bonded two-phase composite free of cracks or voids.

The composite material is assumed to consist of elastic fibers and an elastic matrix. The fibers are continuous, aligned and equally spaced. It is also assumed that the elastic strains are small (infinitesimal). Therefore, the elastic strain tensor can be taken to be the usual engineering elastic strain tensor $\boldsymbol{\varepsilon}$. It is also assumed that there exists an elastic strain energy function such that a linear relation can be used between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the engineering elastic strain tensor $\boldsymbol{\varepsilon}$. In fact, the tensor rate $\dot{\boldsymbol{\varepsilon}}$ for small elastic deformations is equal to the elastic part of the spatial strain rate tensor $\mathbf{d}$ where second order terms are neglected.

In the following, quantities are defined in the configuration C of the overall composite system. Barred quantities are defined in the configuration $\bar{C}$ of the overall composite system. Only Cartesian tensors are considered in this work with their tensor components denoted by subscripts with the usual summation convention. Quantities with a superscript M or F refer to matrix or fiber related quantities, respectively. The superscript R is used to indicate the matrix or fibers where no distinction between them is necessary. No summation is assumed between a superscript and the corresponding identical subscript. It follows directly that barred quantities with a superscript M or F (or R in general) refer to marix or fiber related quantities, respectively, in the configuration $\bar{C}$. For example, $\boldsymbol{\sigma}$ is the composite (overall) Cauchy stress in C, $\bar{\boldsymbol{\sigma}}$ is the effective composite Cauchy stress in $\bar{C}$, $\boldsymbol{\sigma}^{\mathrm{M}}$ and $\boldsymbol{\sigma}^{\mathrm{F}}$ are the matrix and fiber stresses in C, respectively, and $\bar{\boldsymbol{\sigma}}^{\mathrm{M}}$ and $\bar{\boldsymbol{\sigma}}^{\mathrm{F}}$ are the effective matrix and fiber stresses in $\bar{C}$, respectively.

The constitutive model is first formulated in the configuration $\bar{C}$ of the composite system. Then the hypothesis of elastic energy equivalence (Sidoroff, 1981) is used to transform the model into the configuration C of the composite system. In this hypothesis, it is assumed that the elastic energy for a damaged material is equivalent in form to that of the undamaged material except that the stress is replaced by the effective stress in the energy formulation. For this purpose, certain transformation equations are derived for the composite (overall) stresses and strains between the configurations C and $\bar{C}$.

In the formulation, the Eulerian reference system is used, i.e. all quantities are based on spatial coordinates. In Section 2.2, the necessary equations relating local and overall quantities of the composite system are presented. The continuum damage mechanics equations are then derived in Section 2.3. Then the constitutive equations are derived in Section 2.4.

2.2. *Composite analysis*

In this section, the relations between the local (matrix and fiber) and overall (composite) relations are presented in the configuration $\bar{C}$. The analysis is based on the model given by Dvorak and Bahei-El-Din (1982, 1987) and Bahei-El-Din and Dvorak (1989) utilizing a representative volume element that is statistically homogeneous with uniform overall fields of stress or strain. In this case, the composite system consists of an elastic metal matrix reinforced by elastic, continuous aligned fibers.

In the configuration $\bar{C}$, the effective stress tensor $\bar{\boldsymbol{\sigma}}^{\mathrm{R}}$ is related to the effective composite stress tensor $\bar{\boldsymbol{\sigma}}$ by

$$\bar{\sigma}_{ij}^{\mathrm{R}} = B_{ijkl}^{\mathrm{R}} \bar{\sigma}_{kl}, \tag{1}$$

where B_{ijkl}^{R} is a fourth-rank tensor indicating the elastic phase stress concentration factor and the superscript R stands for either M or F. The tensor $\mathbf{B}^{\mathrm{R}}(\mathbf{x})$ depends only on the spatial coordinates $\mathbf{x}$ for the case of elastic deformation. In order to determine $\mathbf{B}^{\mathrm{R}}$, certain assumptions are employed, like the Voigt assumption where the matrix and fibers are assumed to deform equally or the VFD assumption where the fibers are assumed to have vanishing diameters while occupying a finite volume fraction. These two assumptions are discussed briefly among others at the end of Section 2.4. The reader should note that the tensor $\mathbf{B}^{\mathrm{R}}$ does not include any damage effect. This is the reason why effective stresses are used in eqn (1) rather than the actual stresses. In the sequel, a damage phase stress concentration factor will be derived in terms of $\mathbf{B}^{\mathrm{R}}$ and the damage variables.

As a result of volume integration and averaging of the local stress fields, the following relation is obtained between the local (matrix and fiber) stresses and the overall stress in $\bar{C}$:

$$\bar{\sigma}_{ij} = c^{\mathrm{M}} \bar{\sigma}_{ij}^{\mathrm{M}} + c^{\mathrm{F}} \bar{\sigma}_{ij}^{\mathrm{F}}, \tag{2}$$

where c^{M} and c^{F} are the matrix and fiber volume fractions, respectively, given by:

$$c^{\mathrm{R}} = V^{\mathrm{R}}/V, \quad V = V^{\mathrm{M}} + V^{\mathrm{F}}. \tag{3}$$

In eqn (3), V^{M} and V^{F} are the matrix and fiber volumes, respectively, and V is the total volume of the representative composite element. Using the assumption in eqn (2) and substituting the relevant expressions for $\bar{\boldsymbol{\sigma}}^{\mathrm{M}}$ and $\bar{\boldsymbol{\sigma}}^{\mathrm{F}}$ from eqn (1), one derives the following relation between the elastic stress concentration factors for the matrix and fibers:

$$c^{\mathrm{M}} B_{ijkl}^{\mathrm{M}} + c^{\mathrm{F}} B_{ijkl}^{\mathrm{F}} = \tfrac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}), \tag{4}$$

where δ_{ij} is the Kronecker delta.

It is now seen that once the elastic matrix stress concentration factor $\mathbf{B}^{\mathrm{M}}$ is determined, one can use eqn (4) to find the corresponding fiber stress concentration factor $\mathbf{B}^{\mathrm{F}}$. It also follows from the symmetry of the stress tensor and eqn (1) that the stress concentration factors $\mathbf{B}^{\mathrm{M}}$ and $\mathbf{B}^{\mathrm{F}}$ are symmetric in the sense $B_{ijkl}^{\mathrm{R}} = B_{jikl}^{\mathrm{R}}$. Although tensors $\mathbf{B}^{\mathrm{M}}$ and $\mathbf{B}^{\mathrm{F}}$ have other symmetries, this is the only one needed in the derivations that follow.

Next the effective matrix and fiber deviatoric stresses $\bar{\tau}^{M}$ and $\bar{\tau}^{F}$, respectively, are directly derived from eqn (1) as follows:

$$\bar{\tau}_{ij}^{R} = F_{ijkl}^{R}\bar{\sigma}_{kl}, \tag{5}$$

where the fourth-rank tensor $\mathbf{F}^{R}$ is given by:

$$F_{ijkl}^{R} = B_{ijkl}^{R} - \tfrac{1}{3}B_{ppkl}^{R}\delta_{ij}. \tag{6}$$

Using the relation (6) above, one can derive the following two useful identities:

$$F_{rrkl}^{R} = 0, \tag{7}$$

$$F_{ijkl}^{R}F_{ijmn}^{R} = B_{ijkl}^{R}F_{ijmn}^{R}. \tag{8}$$

In the next part of this section, the local–overall relationships for the effective strain tensor $\bar{\varepsilon}$ in the configuration $\bar{C}$ are presented. Upon volume integrating and averaging the local stress fields (Dvorak and Bahei-El-Din, 1982, 1987), the following local–overall relation is obtained for the effective spatial strain tensor:

$$\bar{\varepsilon}_{ij} = c^{M}\bar{\varepsilon}_{ij}^{M} + c^{F}\bar{\varepsilon}_{ij}^{F}, \tag{9}$$

where the appropriate relations for the effective matrix and fiber strain tensors are used as follows:

$$\bar{\varepsilon}_{ij}^{R} = A_{ijkl}^{R}\bar{\varepsilon}_{kl}, \tag{10}$$

where A_{ijkl}^{R} is a fourth-rank tensor denoting the elastic phase strain concentration factor. The same remarks outlined earlier about the tensor $\mathbf{B}^{R}$ apply again to the tensor $\mathbf{A}^{R}$.

Substituting eqn (10) into (9), one derives the following relation between the elastic matrix and fiber strain concentration factors:

$$c^{M}A_{ijkl}^{M} + c^{F}A_{ijkl}^{F} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{11}$$

It is now clear that once one of the elastic strain concentration factors is determined, eqn (11) can be used to determine the other one.

On the other hand, one may start with the quantity $\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}$ and expand it using eqns (2) and (9). Then one substitutes for the local stresses and strains from eqns (1) and (11) and simplifies to obtain:

$$(c^{M}B_{ijkl}^{M} + c^{F}B_{ijkl}^{F})(c^{M}A_{ijmn}^{M} + c^{F}A_{ijmn}^{F}) = \tfrac{1}{2}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{ml}). \tag{12}$$

The above equation represents the relation between the stress and strain concentration factors for the matrix and the fibers. In view of the relations (5) and (12), it is clear that eqn (12) reduces to an identity. It should be mentioned that once the stress concentration factors B_{ijkl}^{M} and B_{ijkl}^{F} are determined, one can use eqn (12) to find a constraint relation between the strain concentration factors A_{ijkl}^{M} and A_{ijkl}^{F}.

2.3. *Damage analysis*

There are two steps that can be followed in order to develop a continuum damage model for a composite system consisting of fibers and a matrix. First, one considers damage in the overall composite system as a whole continuum. At this step, the model will reflect various types of damage mechanisms such as void growth and coalescence in the matrix, fiber fracture, debonding and delamination, etc. It should be noted that at this step, no distinction is made between these types of damage as they are all reflected through the fourth-rank overall damage effect tensor M_{ijkl}. In the second step, one considers the damage that the matrix and fibers undergo separately such as nucleation and growth of voids and void coalescence for the matrix and fracture for the fibers. In this case, two fourth-rank matrix and fiber damage effect tensors M^{M}_{ijkl} and M^{F}_{ijkl} are introduced that reflect all types of damage that the matrix and fibers undergo. Subsequently, the local–overall relations are used to transform these local damage effects to the whole composite system. Therefore, it is clear that the second step does not account explicitly for such damage mechanisms as debonding or delamination. It is also clear that each step has certain advantages and disadvantages. While the first step accounts for all types of damage in the composite system, it cannot distinguish between them. In contrast the second step provides separate damage analysis of the matrix and fiber material but lacks the ability to account for fiber–matrix interaction damage. Therefore, the aim of the proposed model will be to combine the two aforementioned steps in such a way so as to isolate the various local types of damage.

Following the first step outlined above and utilizing an overall damage effect tensor $\mathbf{M}$ for the whole composite system, the overall effective Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ is given by

$$\bar{\sigma}_{ij} = M_{ijkl}\sigma_{kl}. \tag{13}$$

The above relation was first proposed for the uniaxial case by Kachanov (1958) and later generalized to three dimensions by Murakami (1988) and Sidoroff (1981) in the framework of the concept of effective stress. It then follows from eqn (13) that the overall effective deviatoric Cauchy stress tensor $\bar{\boldsymbol{\tau}}$ is given by (Kattan and Voyiadjis, 1990; Voyiadjis and Kattan, 1992)

$$\bar{\tau}_{ij} = N_{ijkl}\sigma_{kl}, \tag{14}$$

where the fourth-rank tensor $\mathbf{N}$ is given in terms of $\mathbf{M}$ as follows:

$$N_{ijkl} = M_{ijkl} - \tfrac{1}{3}M_{rrkl}\delta_{ij}. \tag{15}$$

Certain useful identities follow directly from eqn (15). The main two identities used here are listed below:

$$N_{rrkl} = 0, \tag{16}$$

$$N_{ijkl}N_{ijmn} = M_{ijkl}N_{ijmn}. \tag{17}$$

Next, the relation between the effective phase stress tensor $\bar{\boldsymbol{\sigma}}^{\mathrm{R}}$ and the overall stress tensor $\boldsymbol{\sigma}$ is derived. This is done by substituting eqn (13) into eqn (1). Therefore, one obtains:

$$\bar{\sigma}^{\mathrm{R}}_{ij} = K^{\mathrm{R}}_{ijkl}\sigma_{kl}, \tag{18}$$

where the fourth-rank tensor K^{R}_{ijkl} is given by:

$$K^{\mathrm{R}}_{ijkl} = B^{\mathrm{R}}_{ijmn}M_{mnkl}. \tag{19}$$

From the symmetry of B^{R}_{ijmn} discussed earlier, it follows from eqn (19) that the tensor $\mathbf{K}^{\mathrm{R}}$ is symmetric in the sense $K^{\mathrm{R}}_{ijkl} = K^{\mathrm{R}}_{jikl}$. It should be noted that the tensor $\mathbf{K}^{\mathrm{R}}$ has other symmetries, but these are not needed in the derivations.

Substituting eqn (13) into eqn (5), one obtains the following expressions for the effective matrix and fiber deviatoric stress tensors:

$$\bar{\tau}^{\mathrm{R}}_{ij} = R^{\mathrm{R}}_{ijkl}\sigma_{kl}, \tag{20}$$

where the fourth-rank tensor $\mathbf{R}^{\mathrm{R}}$ is given by:

$$R^{R}_{ijkl} = F^{R}_{ijmn}M_{mnkl}. \quad (21)$$

Upon examining eqns (5), (6), (20) and (21), one concludes that the tensors $\mathbf{F}^{R}$ and $\mathbf{R}^{R}$ are symmetric in the sense $F^{R}_{ijkl} = F^{R}_{jikl}$ and $R^{R}_{ijkl} = R^{R}_{jikl}$. Furthermore, by substituting eqn (6) into eqn (21) and using eqn (19), one can derive the following relation between the tensors $\mathbf{K}^{R}$ and $\mathbf{R}^{R}$:

$$R^{R}_{ijkl} = K^{R}_{ijkl} - \tfrac{1}{3}K^{R}_{rrkl}\delta_{ij}. \quad (22)$$

The tensor $\mathbf{R}^{M}$ satisfies the two identities discussed earlier, namely $R^{R}_{rrkl} = 0$ and $R^{R}_{ijkl}R^{R}_{ijmn} = K^{R}_{ijkl}R^{R}_{ijmn}$. All these tensors have other symmetries but are not needed in the derivations.

The overall damage relations for the composite system have now been presented in eqns (13)–(22). The overall damage effect tensor $\mathbf{M}$ has been introducd to represent all types of damage that the system undergoes. Following the second step discussed at the beginning of this section, one introduces a phase (local) damage effect tensor $\mathbf{M}^{R}$ that represents the damage mechanisms in the phase material like nucleation, growth and coalescence of voids for the matrix, and fracture of fibers. Therefore, the following local transformation equation is assumed to hold for the phase stress tensor

$$\bar{\sigma}^{R}_{ij} = M^{R}_{ijkl}\sigma^{M}_{kl}. \quad (23)$$

It now follows directly from eqn (23) that

$$\bar{\tau}^{R}_{ij} = N^{R}_{ijkl}\sigma^{R}_{kl}, \quad (24)$$

where the fourth-rank tensor $\mathbf{N}^{R}$ satisifes the relations in eqns (15), (16) and (17). Comparing eqns (18) and (23) and simplifying, one derives the following relation between the phase stress tensor and the overall stress tensor:

$$\sigma^{R}_{ij} = \bar{B}^{R}_{ijkl}\sigma_{kl}, \quad (25)$$

where

$$\bar{B}^{R}_{ijkl} = (M^{R}_{mnij})^{-1}B^{R}_{mnpq}M_{pqkl}, \quad (26)$$

where the inverse W^{-1}_{ijkl} of a fourth-rank tensor W_{ijkl} is defined by $W_{ijmn}W^{-1}_{mnkl} = \delta_{ik}\delta_{jl}$. The fourth-rank tensor $\bar{\mathbf{B}}^{R}$ is the damaged phase stress concentration factor that includes geometrical and damage related effects as can be seen from eqn (26).

It is now possible to derive the required relationship between the local damage effect tensors $\mathbf{M}^{M}$ and $\mathbf{M}^{F}$ and the overall damage effect tensor $\mathbf{M}$. Substituting eqn (23) into eqn (2) and simplifying, one obtains the desired relation:

$$M_{ijkl} = c^{M}M^{M}_{ijmn}\bar{B}^{M}_{mnkl} + c^{F}M^{F}_{ijmn}\bar{B}^{F}_{mnkl}. \quad (27)$$

It is clear that eqn (27) relates the local damage experienced by the matrix and fibers to the overall damage of the composite system. The damaged matrix and fiber concentration factors appear in the equation as well as the matrix and fiber volume fractions. Substituting for $\bar{\mathbf{B}}^{M}$ and $\bar{\mathbf{B}}^{F}$ from eqn (26) into eqn (27), one obtains:

$$M_{ijkl} = c^{M}B^{M}_{ijmn}M_{mnkl} + c^{F}B^{F}_{ijmn}M_{mnkl}. \quad (28)$$

The above equation is an explicit relation between the effective local concentration factors and the overall damage effect tensor. Examining eqns (27) and (28) carefully, one concludes that once the local (matrix and fiber) damage mechanisms have been described through the tensors $\mathbf{M}^{M}$ and $\mathbf{M}^{F}$, then the overall damage in the composite system can be described which includes the matrix and fiber related damage as well as the damage resulting from the interaction of the two phases such as debonding.

2.4. *Constitutive equations*

In this section, the elastic constitutive relation for the damaged composite system will be developed. First, one starts with the overall system and assumes the material obeys generalized Hooke's law in the undamaged configuration $\bar{C}$:

$$\bar{\sigma}_{ij} = E_{ijkl}\bar{\varepsilon}_{kl}, \tag{29}$$

where E_{ijkl} is the constant fourth-rank elasticity tensor. The corresponding effective elastic strain energy $\bar{U}$ in this configuration is given by

$$\bar{U} = \tfrac{1}{2}E_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}. \tag{30}$$

One now uses the Legendre transform and applies it to eqn (30) in order to derive the following expression for the effective elastic complementary strain energy $\bar{V}$:

$$\bar{V} = \tfrac{1}{2}E^{-1}_{ijkl}\bar{\sigma}_{ij}\bar{\sigma}_{kl}. \tag{31}$$

In the damaged composite configuration C, the elastic constitutive relation takes the form

$$\sigma_{ij} = \bar{E}_{ijkl}\varepsilon_{kl}, \tag{32}$$

where the fourth-rank tensor $\bar{E}_{ijkl}$ is no longer constant but depends on the damage effect tensor M_{ijkl}. Using the hypothesis of elastic energy equivalence, by equating the energy in eqns (34) in both the damaged and undamaged configurations, i.e. $\bar{V} = V$, one derives the following expression for $\bar{E}_{ijkl}$ (Kattan and Voyiadjis, 1990):

$$\bar{E}_{ijkl} = M^{-1}_{pqkl}E_{rspq}M^{-1}_{rsij}. \tag{33}$$

Differentiating eqn (31) with respect to $\bar{\boldsymbol{\sigma}}$, and using eqns (13) and (33), one can obtain the following transformation equation for the overall strain tensor in the configuration $\bar{C}$:

$$\bar{\varepsilon}_{ij} = M^{-1}_{ijpq}\varepsilon_{pq}. \tag{34}$$

The above eqns (29)–(34) are a brief review of the elastic constitutive relations for a damaged one-phase material and can be used as the overall relations for the composite system. Next, one considers the local stresses and strains in an attempt to formulate local–overall equations for the damaged composite system.

Similar relations can be shown to exist on the local level, that is, the strain transformation equations for the matrix and fibers are similar to eqn (34) and take the following form:

$$\bar{\varepsilon}^{R}_{ij} = (M^{R}_{ijmn})^{-1}\varepsilon^{R}_{mn}. \tag{35}$$

One can now derive expressions for the damaged strain concentration factors. Substituting the expressions of the transformation of the overall and local strains of eqns (34) and (35) into eqn (10) and simplifying, one obtains

$$\varepsilon^{R}_{ij} = \bar{A}^{R}_{ijkl}\varepsilon_{kl}, \tag{36}$$

where the damaged strain concentration factor $\bar{\mathbf{A}}^{R}$ is given by:

$$\bar{A}^{R}_{ijkl} = M^{R}_{mnij} A^{R}_{mnpq} M^{-1}_{pqkl}. \tag{37}$$

One then assumes the generalized Hooke's law to hold for each of the phases in the configuration $\bar{C}$, that is

$$\bar{\sigma}^{R}_{ij} = E^{R}_{ijkl} \bar{\varepsilon}^{R}_{kl}, \tag{38}$$

where E^{R}_{ijkl} is the constant elasticity tensor for the phase material. Substituting eqns (29) and (38) into eqn (2) along with eqns (10) and (11) and simplifying, one obtains the following local–overall relation for the undamaged elasticity tensors:

$$E_{ijkl} = c^{M} A^{M}_{ijmn} E^{M}_{mnkl} + c^{F} A^{F}_{ijmn} E^{F}_{mnkl}. \tag{39}$$

Assuming similar local relations to hold as those of eqn (38) in the configuration C, one has

$$\sigma^{R}_{ij} = \bar{E}^{R}_{ijkl} \varepsilon^{R}_{kl}, \tag{40}$$

where the tensor $\bar{E}^{R}_{ijkl}$ is the damaged elasticity tensor for the phase material. Substituting eqns (32) and (40) into an equation similar to eqn (2), written in the configuration C, along with eqns (35) and simplifying, one obtains

$$\bar{E}_{ijkl} = \bar{c}^{M} \bar{A}^{M}_{ijmn} \bar{E}^{M}_{mnkl} + \bar{c}^{F} \bar{A}^{F}_{ijmn} \bar{E}^{F}_{mnkl}. \tag{41}$$

Next, one considers the transformation equations for the local moduli of elasticity E^{M}_{ijkl} and E^{F}_{ijkl}. Starting with eqn (41) and substituting for $\bar{\mathbf{E}}$ from eqn (33), for $\bar{\mathbf{A}}^{M}$ and $\bar{\mathbf{A}}^{F}$ from eqns (37) and for $\mathbf{E}$ from eqn (39) and simplifying, one obtains:

$$\bar{E}^{R}_{ijkl} = \frac{c^{R}}{\bar{c}^{R}} M^{-1}_{mnij} E^{R}_{mnpq} (M^{R}_{pqkl})^{-1}. \tag{42}$$

The remainder of this section is left to determine a proper transformation relation for the phase volume fractions c^{M} and c^{F}. The authors see no direct way of deriving such equations at the present time. However, in view of the relation given in eqn (33), the scalar ratios $c^{M}/\bar{c}^{M}$ and $c^{F}/\bar{c}^{F}$ can be determined by comparison. Therefore, the following transformation equation is listed here without proof:

$$c^{R} = \frac{\bar{c}^{R}}{9} (M^{R}_{pqij})^{-1} M_{pqij}. \tag{43}$$

Upon substituting the above transformation equation for c^{R} into eqn (42), one obtains simple relations for the transformation of the local moduli of elasticity similar to that of eqn (33). It should be noted that eqn (43) does not imply a change in the phase volume fractions. The quantities $\bar{c}^{M}$ and $\bar{c}^{F}$ represent effective phase volume fractions in the fictitious undamaged configurations $\bar{C}^{M}$ and $\bar{C}^{F}$, respectively. They should be regarded similarly to the effective stresses $\bar{\boldsymbol{\sigma}}^{M}$ and $\bar{\boldsymbol{\sigma}}^{F}$ where they do not represent actual quantities, but effective quantities in the context of continuum damage mechanics. The constitutive theory as well as the relevant transformation equations have been presented for the analysis of damage and small elastic deformation of fiber-reinforced composite materials. This is illustrated in detail in Section 4 for the case of uniaxial tension.

The rest of this section is left for a brief discussion of the stress and strain concentration factors. The simplest model available involves the Voigt assumption where the matrix, fiber and overall strain rates are assumed equal. In our case, the Voigt model is applied to the configuration $\bar{C}$ in the form $\bar{\varepsilon}^{M}_{ij} = \bar{\varepsilon}^{F}_{ij} = \bar{\varepsilon}_{ij}$. Substituting these into eqn (10) immediately results in $A^{M}_{ijkl} = A^{F}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Using this result along with eqns (1), (29) and

(38) will yield $B^{M}_{ijkl} = E^{M}_{ijmn}E^{-1}_{klmn}$ and $B^{F}_{ijkl} = E^{F}_{ijmn}E^{-1}_{klmn}$. Another widely known model is the Vanishing Fiber Diameter (VFD) model. In this model, it is assumed that the cylindrical fibers have vanishing diameters while occupying a finite volume fraction of the composite. However, the resulting equations are not as simple as those of the Voigt model and the reader is referrred to Dvorak and Bahei-El-Din (1979, 1982) for a detailed discussion of the VFD model. Finally, it should be noted that more sophisticated models (Mori and Tanaka, 1973) can also be used.

3. EVOLUTION OF DAMAGE

In order to study the evolution of damage in composite materials, one first needs to investigate the nature of the fourth-rank damage effect tensor **M**. It has been shown (Voyiadjis and Kattan, 1992) that using the Voigt notation for stresses and strains (i.e. representing them as vectors instead of tensors), the tensor M_{ijkl} can be represented by a 6×6 matrix in terms of a second-rank damage tensor $\boldsymbol{\phi}$. Therefore, it is clear (Voyiadjis and Kattan, 1992) that the study of damage evolution involves the determination of an appropriate kinetic equation for the tensor ϕ_{ij}. One introduces the generalized thermodynamic force y_{ij} that is associated with ϕ_{ij} by the definition (Lemaitre, 1985):

$$y_{ij} = \frac{\partial U}{\partial \phi_{ij}}, \tag{44}$$

such that $\dot{\phi}_{ij}y_{ij}$ is the power dissipated due to the damage. The criterion for damage evolution used here is that proposed by Lee *et al.* (1985) and is given by the function $g(\mathbf{y}, B)$ defined by:

$$g(\mathbf{y}, B) = \tfrac{1}{2}J_{ijkl}y_{ij}y_{kl} - B(\beta) = 0, \tag{45}$$

where $B(\beta)$ is a function of the overall damage parameter β and J_{ijkl} is a constant fourth-rank tensor that can be represented by a constant 6×6 matrix (Lee *et al.*, 1985; Voyiadjis and Kattan, 1992). In order to develop an evolution equation for the damage variable $\boldsymbol{\phi}$, one considers the power of dissipation Π given by:

$$\Pi = -y_{pq}\dot{\phi}_{pq} - B\dot{\beta}. \tag{46}$$

The problem is to extremize Π subject to the constraint $g = 0$. Therefore, one introduces the Lagrange multiplier $\dot{\lambda}$ and uses the Lagrange multiplier method to obtain $\dot{\lambda} = \dot{\beta}$ and

$$\dot{\phi}_{pq} = -\dot{\beta}\frac{\partial g}{\partial y_{pq}}. \tag{47}$$

In order to determine $\dot{\beta}$, one uses the consistency condition $\dot{g} = 0$ in the form

$$\frac{\partial g}{\partial y_{pq}}\dot{y}_{pq} + \frac{\partial g}{\partial B}\dot{B} = 0. \tag{48}$$

Substituting for the partial derivatives of g from eqn (45) into eqn (48) and solving for $\dot{\beta}$, one obtains:

$$\dot{\beta} = \dot{\lambda} = \frac{J_{pqmn}y_{mn}\dot{y}_{pq}}{\partial B/\partial \beta}. \tag{49}$$

Substituting the above expression of $\dot{\beta}$ into eqn (47), one obtains the required evolution equation for the damage tensor ϕ_{ij}:

$$\dot{\phi}_{ij} = \frac{-J_{pqmn}\dot{y}_{mn}y_{pq}}{\partial B/\partial \beta}\frac{\partial q}{\partial y_{ij}}. \tag{50}$$

The solution of the above kinetic equation hinges on the determination of an appropriate expression for the function $B(\beta)$. One may use a linear function in the form $B(\beta) = c_1\beta + c_2$ where c_1 and c_2 are constants. This is motivated by analogy to the isotropic hardening parameter κ in the theory of plasticity, where the evolution of κ is taken to be $\dot{\kappa} = (\dot{\varepsilon}''_{ij}\dot{\varepsilon}''_{ij})^{1/2}$ where $\dot{\varepsilon}''_{ij}$ is the rate of plastic strain. Analogously, using $\dot{B} = (\dot{\beta}\dot{\beta})^{1/2}$ will yield a linear function $B(\beta)$ as proposed above. An example is given in the next section where the evolution equation is solved for the case of uniaxial tension with a linear function $B(\beta)$.

4. EXAMPLE 1: UNIAXIAL TENSION OF A UNIDIRECTIONAL LAMINA

Consider a unidirectional fiber-reinforced thin lamina that is subjected to a uniaxial tensile force T along the x_1-direction as shown in Fig. 1. The lamina is made of an elastic matrix with elastic fibers aligned along the x_1-direction. The x_2- and x_3-axes are assumed to lie in the plane of the lamina. Let dS be the cross-sectional area of the lamina with dS^M and dS^F being the cross-sectional areas of the matrix and fibers, respectively. In the fictitious undamaged configuration, let the cross-sectional areas of the lamina, matrix and fibers be denoted by $d\bar{S}$, $d\bar{S}^M$ and $d\bar{S}^F$, respectively. Since the lamina strictly consists of a matrix and fibers, it is clear that $dS^M + dS^F = dS$, $d\bar{S}^M + d\bar{S}^F = d\bar{S}$, $d\bar{S} \leqslant dS$, $d\bar{S}^M \leqslant dS^M$ and $d\bar{S}^F \leqslant dS^F$ (Kattan and Voyiadjis, 1992).

The overall stress, strain and damage tensors $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\phi}$ for this problem can be represented using the following vectors:

$$\boldsymbol{\sigma} \equiv \begin{Bmatrix} \sigma \\ 0 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix}, \quad \boldsymbol{\phi} \equiv \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}, \tag{51}$$

with similar vector representations for their corresponding effective and local counterparts. The uniaxial stress σ appearing in eqn (51) is clearly given by $\sigma = T/dS$ with the uniaxial

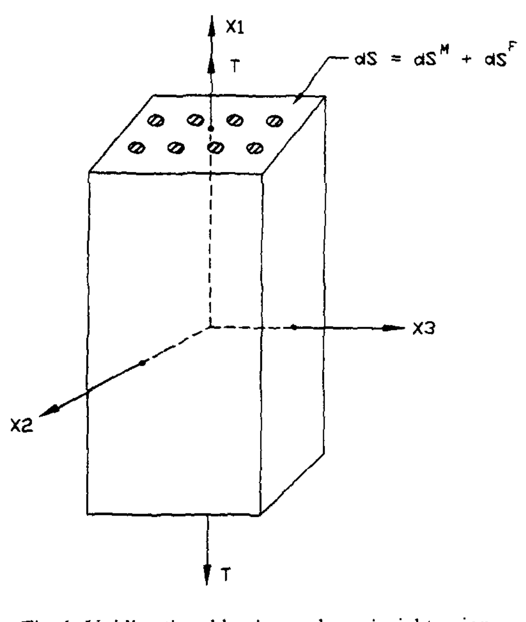

Fig. 1. Unidirectional lamina under uniaxial tension.

effective stress $\bar{\sigma}$ given by $\bar{\sigma} = T/\mathrm{d}\bar{S}$. The overall damage variable ϕ_1 is defined by (Kachanov, 1958)

$$\phi_1 = \frac{\mathrm{d}S - \mathrm{d}\bar{S}}{\mathrm{d}S}. \tag{52}$$

It is clear from eqn (52) that ϕ_1 takes the values between 0 for undamaged material to 1 for (theoretically) complete rupture. However, the actual value ϕ_{cr} where failure occurs is less than 1 and satisfies $0 \leqslant \phi < \phi_{cr} < 1$. Two local damage variables ϕ_1^M and ϕ_1^F can be analogously introduced and defined by:

$$\phi_1^R = \frac{\mathrm{d}S^R - \mathrm{d}\bar{S}^R}{\mathrm{d}S^R}. \tag{53}$$

It follows directly from eqn (53) that $0 \leqslant \phi_1^R \leqslant 1$. Using eqns (52) and (53) along with the area relations discussed in the beginning of this section, one can easily derive the following relation between the local and overall damage variables:

$$\phi_1 = c^M \phi_1^M + c^F \phi_1^F. \tag{54}$$

It should be mentioned that the uniaxial local and overall stresses σ, σ^M and σ^F satisfy a similar relation to that of eqn (54) and is given in tensor form by eqn (2). The relation between the overall stress σ and its effective counterpart $\bar{\sigma}$ can be easily shown to be given by

$$\bar{\sigma} = \frac{\sigma}{1-\phi_1}. \tag{55}$$

Using eqn (2) and a similar equation for the effective stress, one can assume the local stresses to be given by:

$$\bar{\sigma}^R = \frac{\sigma^R}{1-\phi_1^R}. \tag{56}$$

In view of eqn (55), it is clear that the relations (56) satisfy the requirements given by eqn (2). Comparing eqns (55) and (56) with the general transformation equations (13) and (23) and considering the notation of eqn (51) for this problem, the damage effect tensors **M**, $\mathbf{M}^M$ and $\mathbf{M}^F$ can be represented by the following matrices:

$$\mathbf{M} \equiv \begin{bmatrix} \frac{1}{1-\phi_1} & 0 & 0 \\ 0 & \frac{1}{1-\phi_2} & 0 \\ 0 & 0 & \frac{1}{1-\phi_3} \end{bmatrix}, \tag{57}$$

$$\mathbf{M}^R \equiv \begin{bmatrix} \frac{1}{1-\phi_1^R} & 0 & 0 \\ 0 & \frac{1}{1-\phi_2^R} & 0 \\ 0 & 0 & \frac{1}{1-\phi_3^R} \end{bmatrix}. \tag{58}$$

It should be mentioned that the matrix representation of the damage effect tensor **M** of eqn (57) applies only to the problem of uniaxial tension considered here. For a general matrix representation of the tensor **M**, the reader is referred to the recent paper by Voyiadjis and Kattan (1992).

The overall elasticity tensor E_{ijkl} can be represented by the following matrix where an orthotropic material is assumed:

$$\mathbf{E}^{-1} \equiv \begin{bmatrix} \dfrac{1}{E_1} & \dfrac{-\nu_{21}}{E_2} & \dfrac{-\nu_{31}}{E_3} \\ \dfrac{-\nu_{12}}{E_1} & \dfrac{1}{E_2} & \dfrac{-\nu_{32}}{E_3} \\ \dfrac{-\nu_{13}}{E_1} & \dfrac{-\nu_{23}}{E_2} & \dfrac{1}{E_3} \end{bmatrix}. \tag{59}$$

Using the representations of **M** and **E** in eqns (57) and (59), and substituting them into the transformation equation (33), one obtains the following matrix for the damaged elasticity tensor $\bar{E}_{ijkl}$:

$$\bar{\mathbf{E}}^{-1} \equiv \begin{bmatrix} \dfrac{1}{E_1(1-\phi_1)^2} & \dfrac{-\nu_{21}}{E_2(1-\phi_1)(1-\phi_2)} & \dfrac{-\nu_{31}}{E_3(1-\phi_1)(1-\phi_3)} \\ \dfrac{-\nu_{12}}{E_1(1-\phi_1)(1-\phi_2)} & \dfrac{1}{E_2(1-\phi_2)^2} & \dfrac{-\nu_{32}}{E_3(1-\phi_2)(1-\phi_3)} \\ \dfrac{-\nu_{13}}{E_1(1-\phi_1)(1-\phi_3)} & \dfrac{-\nu_{23}}{E_2(1-\phi_2)(1-\phi_3)} & \dfrac{1}{E_3(1-\phi_3)^2} \end{bmatrix}. \tag{60}$$

Considering a matrix representation for $\bar{\mathbf{E}}^{-1}$ similar to that of eqn (59) but with all quantities replaced by barred quantities and comparing it with the matrix in eqn (60), one obtains the following transformation equations for the overall elastic properties:

$$\bar{E}_i = E_i(1-\phi_i)^2, \quad i = 1,2,3 \text{ (no sum)}, \tag{61}$$

$$\bar{\nu}_{ij} = \nu_{ij}\frac{1-\phi_i}{1-\phi_j}, \quad i,j = 1,2,3 \text{ (no sum)}. \tag{62}$$

Next, one uses the transformation equation (26) for the phase stress concentration factors, and substitutes for the damage effect tensors from eqns (57) and (58) to derive the following matrix representation for the damage phase stress concentration factor $\bar{B}^{R}_{ijkl}$:

$$\bar{\mathbf{B}}^{R} \equiv \begin{bmatrix} \dfrac{1-\phi^R_1}{1-\phi_1}B^R_{11} & \dfrac{1-\phi^R_1}{1-\phi_2}B^R_{12} & \dfrac{1-\phi^R_1}{1-\phi_3}B^R_{13} \\ \dfrac{1-\phi^R_2}{1-\phi_1}B^R_{21} & \dfrac{1-\phi^R_2}{1-\phi_2}B^R_{22} & \dfrac{1-\phi^R_2}{1-\phi_3}B^R_{23} \\ \dfrac{1-\phi^R_3}{1-\phi_1}B^R_{31} & \dfrac{1-\phi^R_3}{1-\phi_2}B^R_{32} & \dfrac{1-\phi^R_3}{1-\phi_3}B^R_{33} \end{bmatrix}, \tag{63}$$

where the terms B^R_{ij} are the elements of the matrix representation of B^R_{ijkl}.

Similarly, one uses the transformation equation (37) for the strain concentration factors to derive the following matrix representation for the damaged phase strain concentration factor $\bar{A}^{R}_{ijkl}$:

$$\bar{\mathbf{A}}^{\mathrm{R}} \equiv \begin{bmatrix} \frac{1-\phi_1}{1-\phi_1^{\mathrm{R}}} A_{11}^{\mathrm{R}} & \frac{1-\phi_1}{1-\phi_2^{\mathrm{R}}} A_{12}^{\mathrm{R}} & \frac{1-\phi_1}{1-\phi_3^{\mathrm{R}}} A_{13}^{\mathrm{R}} \\ \frac{1-\phi_2}{1-\phi_1^{\mathrm{R}}} A_{21}^{\mathrm{R}} & \frac{1-\phi_2}{1-\phi_2^{\mathrm{R}}} A_{22}^{\mathrm{R}} & \frac{1-\phi_2}{1-\phi_3^{\mathrm{R}}} A_{23}^{\mathrm{R}} \\ \frac{1-\phi_3}{1-\phi_1^{\mathrm{R}}} A_{31}^{\mathrm{R}} & \frac{1-\phi_3}{1-\phi_2^{\mathrm{R}}} A_{32}^{\mathrm{R}} & \frac{1-\phi_3}{1-\phi_3^{\mathrm{R}}} A_{33}^{\mathrm{R}} \end{bmatrix}, \tag{64}$$

where the terms A_{ij}^{R} are the elements in the matrix representation of A_{ijkl}^{R}.

Finally, one writes the transformation equations for the volume fractions c^{M} and c^{F}. Using eqn (43) along with the matrix representations (57) and (58), one derives:

$$\bar{c}^{\mathrm{R}} = c^{\mathrm{R}} \frac{1-\phi_1^{\mathrm{R}}}{1-\phi_1}. \tag{65}$$

Alternatively, the above relations can be derived independently using the definitions of c^{M} and c^{F} as area fractions for this problem, along with eqns (52) and (53). Finally, one can use eqns (42) and (43) to derive transformation equations for the local elastic properties. However, the resulting equations are similar to eqns (61) and (62) with superscripts M or F and will not be listed here.

In order to characterize damage evolution for this problem, one uses eqn (50). For this problem, the kinetic equation (50) reduces to

$$\dot{\phi}_1 = \frac{-y_1^2 \dot{y}_1}{\partial B/\partial \beta}, \tag{66}$$

where ϕ_1 and y_1 stand for the tensor components ϕ_{11} and y_{11}, respectively. Using a linear function $B(\beta) = c_1\beta + c_2$ where c_1 and c_2 are constants, substituting it into eqn (66) and solving the differential equation, one obtains

$$\phi_1 = -\frac{y_1^3}{3c_1}. \tag{67}$$

The above equation represents the relation between the damage variable ϕ_1 and its associated thermodynamic generalized force y_1 for the case of uniaxial tension. One then substitutes for y_1 from eqn (44) along with eqn (30) into eqn (67) to obtain:

$$\frac{\phi_1}{(1-\phi_1)^6} = \frac{E^3}{3c_1}\varepsilon_1^6. \tag{68}$$

Equation (68) represents the overall strain–damage relation for the case of uniaxial tension. Similar relations can be derived for the local strain and damage variables.

5. EXAMPLE 2: A UNIDIRECTIONAL LAMINA UNDER PLANE STRESS

Consider a unidirectional fiber-reinforced thin lamina that is subjected to a case of plane stress in the 1–2 plane as shown in Fig. 2. The lamina is made of an elastic material with elastic fibers aligned along the x_1-axis. Both the x_1- and x_2-axes are assumed to lie in the plane of the lamina, while the x_3-axis lies in the transverse direction to that plane. A complete damage state is considered, although the lamina is under plane stress. Therefore, all the damage variables are assumed nonzero in this example. For this case of plane stress, the stress and damage tensors are represented by the following matrices:

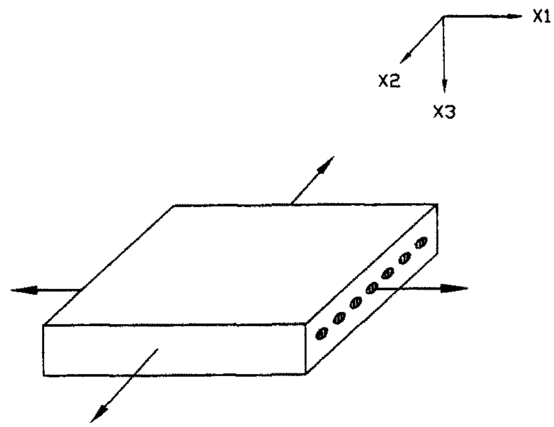

Fig. 2. Unidirectional lamina under plane stress in the 1–2 plane.

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{69}$$

$$[\phi_{ij}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix}. \tag{70}$$

The transformation equation (13) gives rise to a nonsymmetric effective stress tensor. Therefore, before proceeding with this example, one needs to symmetrize the effective stress tensor $\bar{\boldsymbol{\sigma}}$. One of the most popular symmetrization procedures is given by the equation

$$\bar{\sigma}_{ij} = \tfrac{1}{2}[\sigma_{ik}(\delta_{kj}-\phi_{kj})^{-1}+(\delta_{il}-\phi_{il})^{-1}\sigma_{lj}]. \tag{71}$$

Before substituting eqns (69) and (70) into eqn (71), one needs to find the inverse of the tensor $\delta_{ij}-\phi_{ij}$. This is found simply by obtaining the inverse of the matrix $[\delta_{ij}-\phi_{ij}]$ through the use of the symbolic manipulation program REDUCE. The resulting matrix is given as:

$$[\delta_{ij}-\phi_{ij}]^{-1} = \frac{1}{\Delta} \times \begin{bmatrix} (1-\phi_{22})(1-\phi_{33})-\phi_{23}^2 & \phi_{13}\phi_{23}+\phi_{12}(1-\phi_{33}) & \phi_{12}\phi_{23}+\phi_{13}(1-\phi_{22}) \\ \phi_{13}\phi_{23}+\phi_{12}(1-\phi_{33}) & (1-\phi_{11})(1-\phi_{33})-\phi_{13}^2 & \phi_{12}\phi_{13}+\phi_{23}(1-\phi_{11}) \\ \phi_{12}\phi_{23}+\phi_{13}(1-\phi_{22}) & \phi_{12}\phi_{13}+\phi_{23}(1-\phi_{11}) & (1-\phi_{11})(1-\phi_{22})-\phi_{12}^2 \end{bmatrix}, \tag{72}$$

where Δ is given by

$$\Delta = (1-\phi_{11})(1-\phi_{22})(1-\phi_{33})-\phi_{23}^2(1-\phi_{11}) -\phi_{13}^2(1-\phi_{22})-\phi_{12}^2(1-\phi_{33})-2\phi_{12}\phi_{23}\phi_{13}. \tag{73}$$

Next, one substitutes eqns (69) and (73) into eqn (71) and simplies the resulting matrix. Using the vector representation $[\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^T$ for the stress tensor $\boldsymbol{\sigma}$, and likewise using $[\bar{\sigma}_{11} \quad \bar{\sigma}_{22} \quad \bar{\sigma}_{12}]^T$ for the effective stress tensor $\bar{\boldsymbol{\sigma}}$, the resulting equation can be rewritten as:

$$\begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M'_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \tag{74}$$

where the coefficients of the matrix **M** of eqn (74) are given by:

$$M_{11} = \frac{(1-\phi_{22})(1-\phi_{33})-\phi_{23}^2}{\Delta}, \tag{75a}$$

$$M_{22} = \frac{(1-\phi_{11})(1-\phi_{33})-\phi_{13}^2}{\Delta}, \tag{75b}$$

$$M_{33} = \frac{M_{11}+M_{22}}{2}, \tag{75c}$$

$$M_{12} = M_{21} = 0, \tag{75d}$$

$$M_{13} = 2M_{31} = \frac{\phi_{12}\phi_{23}+\phi_{12}(1-\phi_{33})}{\Delta}, \tag{75e}$$

$$M_{23} = 2M_{32} = M_{13}. \tag{75f}$$

The above equations were obtained using the symbolic manipulation program REDUCE.

For the case of plane stress discussed in this section, the overall elasticity tensor E_{ijkl} is written here for an orthotropic material indirectly in the following form:

$$\mathbf{E}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}. \tag{76}$$

Using the representation of **M** and $\mathbf{E}^{-1}$ in eqns (75) and (76) and substituting them into the transformation equation (33), one obtains the following matrix for the damaged elasticity tensor $\bar{E}_{ijkl}$:

$$\bar{\mathbf{E}}^{-1} = \frac{1}{\Delta^2}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}, \tag{77}$$

where the terms in the above matrix are given by:

$$E_{11} = \frac{M_{11}^2}{E_1} + \frac{M_{13}^2}{2G_{12}}, \tag{78a}$$

$$E_{22} = \frac{M_{22}^2}{E_2} + \frac{M_{13}^2}{2G_{12}}, \tag{78b}$$

$$E_{33} = \tfrac{1}{2}M_{13}^2\left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{\nu_{12}}{E_1} - \frac{\nu_{21}}{E_2}\right) + \frac{(M_{11}+M_{22})^2}{4G_{12}}, \tag{78c}$$

$$E_{12} = -\nu_{12}\frac{M_{11}M_{22}}{E_1} + \frac{M_{13}^2}{2G_{12}}, \tag{78d}$$

$$E_{13} = M_{13}\left[\frac{M_{11}}{E_1} - \nu_{12}\frac{M_{11}}{E_1} + \frac{(M_{11}+M_{22})}{2G_{12}}\right], \tag{78e}$$

$$E_{23} = M_{13}\left[\frac{M_{22}}{E_2} - \nu_{21}\frac{M_{22}}{E_2} + \frac{(M_{11}+M_{22})}{2G_{12}}\right]. \tag{78f}$$

The inverse of the matrix in eqn (77) should be equivalent to the effective elasticity tensor $\bar{E}_{ijkl}^{-1}$ given by:

$$\bar{\mathbf{E}}^{-1} = \begin{bmatrix} \frac{1}{\bar{E}_1} & -\frac{\bar{\nu}_{12}}{\bar{E}_1} & 0 \\ -\frac{\bar{\nu}_{21}}{\bar{E}_2} & \frac{1}{\bar{E}_2} & 0 \\ 0 & 0 & \frac{1}{\bar{G}_{12}} \end{bmatrix}. \tag{79}$$

The relations between the elastic constant E_1, E_2, ν_{12}, G_{12} and the effective elastic variables $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$ and $\bar{G}_{12}$ can be obtained by equating eqns (77) and (79). After some lengthy algebraic manipulations one arrives at the following relations:

$$\bar{E}_1 = E_1\frac{\Delta^2}{M_{11}}\left[\frac{M_{11}+M_{22}}{M_{11}^2+M_{11}M_{22}-(1-\nu_{12})M_{13}^2}\right], \tag{80a}$$

$$\bar{E}_2 = E_2\frac{\Delta^2}{M_{22}}\left[\frac{M_{11}+M_{22}}{M_{22}^2+M_{11}M_{22}-(1-\nu_{21})M_{13}^2}\right], \tag{80b}$$

$$\bar{\nu}_{12} = \frac{\nu_{12}(M_{22}^2+M_{11}M_{22})-(1-\nu_{12})M_{13}^2}{M_{11}^2+M_{11}M_{22}-(1-\nu_{12})M_{13}^2}, \tag{80c}$$

$$\bar{G}_{12} = G_{12}\frac{4\Delta^2 M_{11}M_{22}}{(M_{11}+M_{22})^2(M_{11}M_{22}-M_{13}^2)}. \tag{80d}$$

In addition, one obtains the following relation:

$$\nu_{12}E_2 = \nu_{21}E_1. \tag{81}$$

The above relation holds for the damaged composite system and is similar to the usual composite relation for effective quantities $\bar{\nu}_{12}\bar{E}_2 = \bar{\nu}_{21}\bar{E}_1$. It should be noted that in eqn (80a), $\bar{E}_1$ is a function of both E_1 and ν_{12} in addition to the damage variables. Similarly, in eqn (80b), $\bar{E}_2$ depends on E_2, ν_{21} and the damage variables. On the other hand, it is clear from eqns (80c) and (80d) that $\bar{\nu}_{12}$ depends only on ν_{12} and $\bar{G}_{12}$ depends only on G_{12} in addition to the damage variables. An expression for $\bar{\nu}_{21}$ in terms of ν_{21} can be obtained similar to eqn (80c).

When using the principal damage variables ϕ_1, ϕ_2 and ϕ_3, eqns (80) reduce to:

$$\bar{E}_1 = E_1\left(\frac{\Delta}{M_{11}}\right)^2, \tag{82a}$$

$$\bar{E}_2 = E_2\left(\frac{\Delta}{M_{22}}\right)^2, \tag{82b}$$

$$\bar{\nu}_{12} = \nu_{12}\frac{M_{22}}{M_{11}}, \tag{82c}$$

$$\bar{G}_{12} = G_{12}\left(\frac{2\Delta}{M_{11}+M_{22}}\right)^2. \tag{82d}$$

Substituting for M_{11}, M_{22} and Δ from eqns (75) into eqns (82), keeping in mind that $\phi_{13} = \phi_{23} = \phi_{12} = M_{13} = 0$, one obtains:

$$\bar{E}_1 = E_1(1-\phi_1)^2, \tag{83a}$$

$$\bar{E}_2 = E_2(1-\phi_2)^2, \tag{83b}$$

$$\bar{\nu}_{12} = \nu_{12}\frac{1-\phi_1}{1-\phi_2}, \tag{83c}$$

$$\bar{G}_{12} = G_{12}\left[\frac{2(1-\phi_1)^2(1-\phi_2)^2(1-\phi_3)^3}{(1-\phi_1)+(1-\phi_2)}\right]^2. \tag{83d}$$

Equations (83a)–(83c) are similar to eqns (61) and (62). Thus, the expressions involving principal damage variables reduce to those of the uniaxial tension case of the previous example. The only new equation here is eqn (83d) which relates shear quantities that do not exist in the uniaxial tension case.

Equations (83a) and (83b) are easily interpreted by the deterioration of the stiffnesses $\bar{E}_1$ and $\bar{E}_2$ for a damaged composite system. In this case, the stiffness degradation is parabolic as shown in Fig. 3 for $\bar{E}_1$. The influence of damage on Poisson's ratio ν_{12} is shown in Fig. 4. This graph is plotted using eqn (83c) showing the variation of the ratio $\bar{\nu}_{12}/\nu_{12}$ with the principal damage variables ϕ_1 and ϕ_2. It is clearly shown that when $\phi_1 = \phi_2 = 0$, the ratio $\bar{\nu}_{12}/\nu_{12} = 1$. When $\phi_1 = 0$, the ratio ν_{12}/ν_{12} becomes the hyperbola $1/(1-\phi_2)$. When $\phi_2 = 0$, the ratio $\bar{\nu}_{12}/\nu_{12}$ becomes the straight line $1-\phi_1$. On the other hand, when ϕ_1 approaches 1, the value of $\bar{\nu}_{12}$ clearly approaches zero. Also, when ϕ_2 approaches 1, the ratio $\bar{\nu}_{12}/\nu_{12}$ approaches infinity indicating complete rupture.

Figure 5 shows the variation of the shear modulus ratio $\bar{G}_{12}/G_{12}$ with the principal damage variables ϕ_1 and ϕ_2 as given in eqn (83d) for selected values of ϕ_3. In Fig. 5(a), the value of $\phi_3 = 0$ is used in the results. It is clear that for the case of no damage ($\phi_1 = \phi_2 = 0$), the ratio $\bar{G}_{12}/G_{12}$ is equal to 1. However, when $\phi_1 = 0$, the ratio approaches the hyperbola $[2(1-\phi_2)^2/(1+(1-\phi_2))]^2$. A similar hyperbola is obtained in the variable ϕ_1 when $\phi_2 = 0$. It is interesting to note that when either ϕ_1 or ϕ_2 approaches 1 (indicating complete rupture), the value of $\bar{G}_{12}$ becomes identically zero. Therefore, the effective elastic moduli vanish

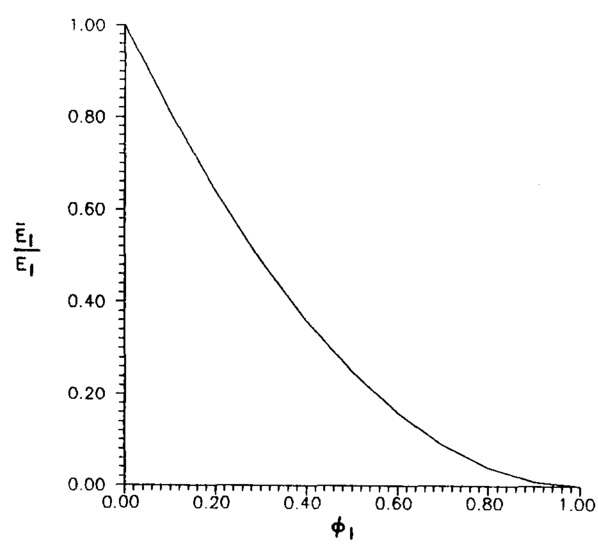

Fig. 3. Variation of $\bar{E}_1/E_1$ with the damage variable ϕ_1.

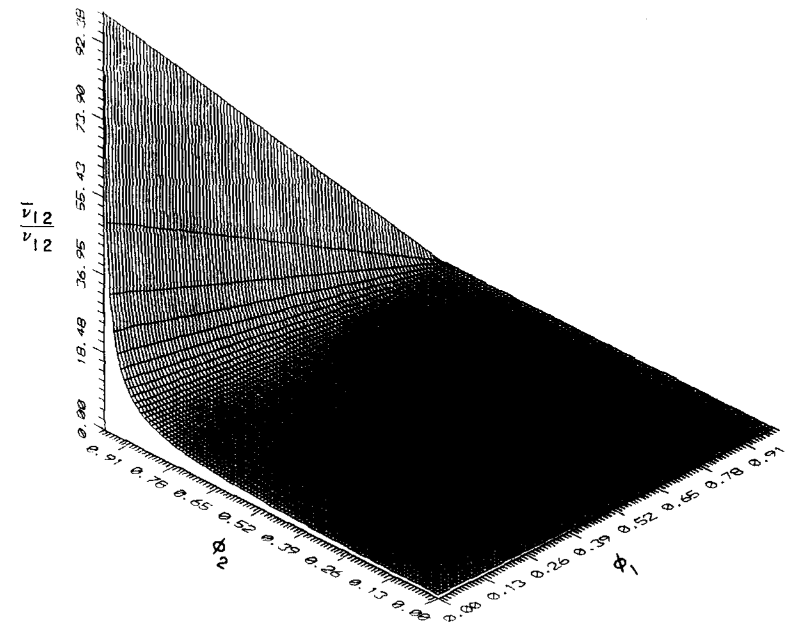

Fig. 4. Variation of $\bar{\nu}_{12}/\nu_{12}$ with the damage variables ϕ_1 and ϕ_2.

when the composite system undergoes complete rupture. Figures 5(b) and 5(c) show the results for values of ϕ_3 of 0.25 and 0.5, respectively.

The phase damage effect tensor M^{R}_{ijkl} is considered here to have the same form as the overall damage effect tensor **M** given in eqns (74) and (75). This tensor takes the form:

$$[M^{R}_{ijkl}] = \frac{1}{\Delta^{R}} \begin{bmatrix} M^{R}_{11} & 0 & M^{R}_{13} \\ 0 & M^{R}_{22} & M^{R}_{13} \\ \frac{1}{2}M^{R}_{13} & \frac{1}{2}M^{R}_{13} & \frac{1}{2}(M^{R}_{11}+M^{R}_{22}) \end{bmatrix}, \tag{84}$$

where Δ^{R}, M^{R}_{11}, M^{R}_{22} and M^{R}_{13} are given by:

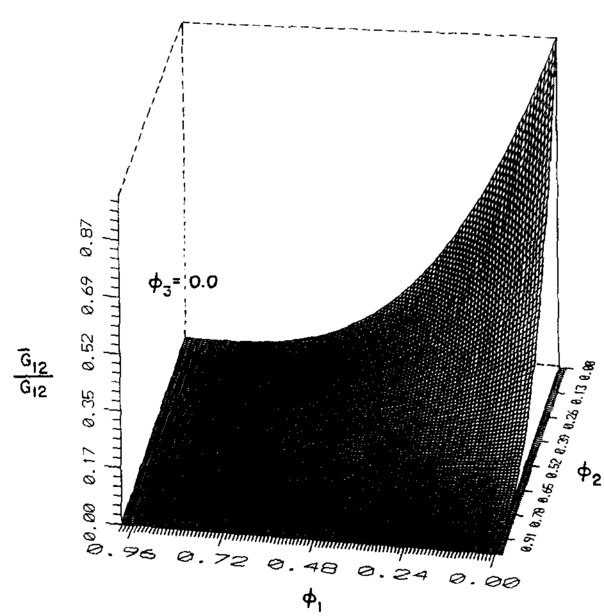

Fig. 5(a). Variation of $\bar{G}_{12}/G_{12}$ with the damage variables ϕ_1 and ϕ_2.

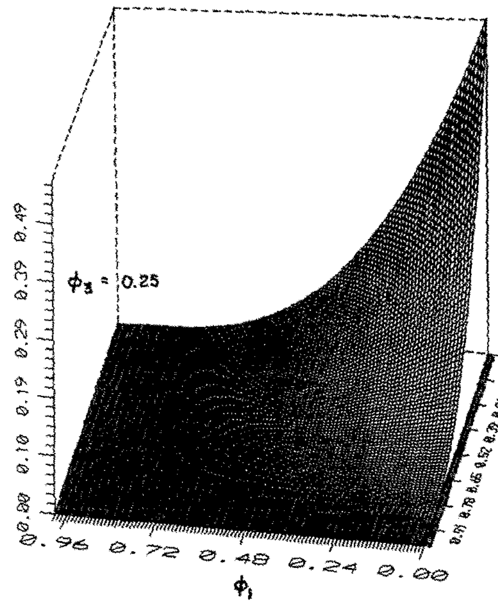

Fig. 5(b). Variation of $\bar{G}_{12}/G_{12}$ with the damage variables ϕ_1 and ϕ_2.

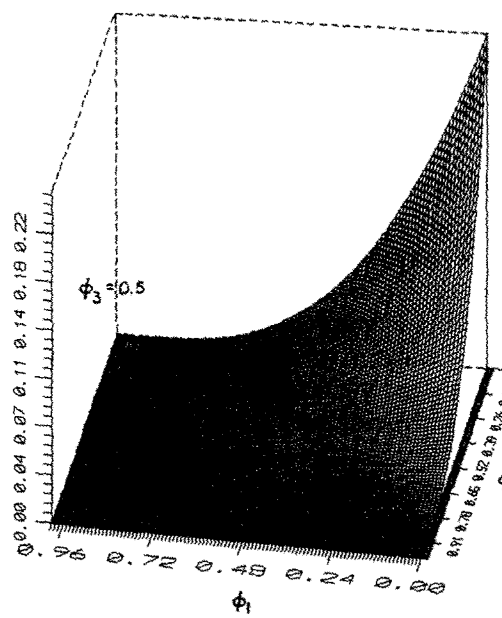

Fig. 5(c). Variation of $\bar{G}_{12}/G_{12}$ with the damage variables ϕ_1 and ϕ_2.

$$\Delta^R = (1-\phi^R_{11})(1-\phi^R_{22})(1-\phi^R_{33})-(\phi^R_{23})^2(1-\phi^R_{11})$$
$$-(\phi^R_{13})^2(1-\phi^R_{22})-(\phi^R_{12})^2(1-\phi^R_{33})-2\phi^R_{12}\phi^R_{23}\phi^R_{13}, \tag{85a}$$

$$M^R_{11} = (1-\phi^R_{22})(1-\phi^R_{33})-(\phi^R_{23})^2, \tag{85b}$$

$$M^R_{22} = (1-\phi^R_{11})(1-\phi^R_{33})-(\phi^R_{13})^2, \tag{85c}$$

$$M^R_{13} = \phi^R_{12}\phi^R_{23}+\phi^R_{12}(1-\phi^R_{33}). \tag{85d}$$

The expressions for $\mathbf{M}^R$ and $\mathbf{M}$ given in eqns (75) and (85), respectively, are substituted into the transformation equation (26) to obtain the matrix and fiber stress concentration factors for the case of plane stress. The phase stress concentration factors $\bar{B}^{R}_{ij}$ are then given by:

$$\begin{Bmatrix} \bar{B}^{R}_{11} \\ \bar{B}^{R}_{22} \\ \bar{B}^{R}_{33} \\ \bar{B}^{R}_{12} \\ \bar{B}^{R}_{13} \\ \bar{B}^{R}_{23} \end{Bmatrix} = L \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{12} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{13} & l_{23} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{bmatrix} \begin{Bmatrix} B^{R}_{11} \\ B^{R}_{22} \\ B^{R}_{33} \\ B^{R}_{12} \\ B^{R}_{13} \\ B^{R}_{23} \end{Bmatrix}, \tag{86}$$

where L is given by:

$$L = \frac{\Delta}{\Delta^{R}[M^{R}_{11}(M^{R}_{22})^{2} + (M^{R}_{11})^{2}M^{R}_{22} - (M^{R}_{13})^{2}(M^{R}_{11} + M^{R}_{22})]}. \tag{87}$$

The coefficients l_{ij} in the above matrix are given in the Appendix and the terms B^{R}_{ij} are the elements of the matrix representation of the tensor B^{R}_{ijkl}.

In the same way, one can use the transformation equation (37) for the strain concentration factors $\bar{A}^{M}_{ijkl}$ and $\bar{A}^{F}_{ijkl}$ to obtain results corresponding to eqn (86). The resulting coefficients look similar to those in eqns (A1)–(A33) of the Appendix and are not shown here for the sake of simplicity.

6. CONCLUSION

A micromechanical model is formulated to study damage in fiber-reinforced composite materials. The composite material is asssumed to consist of an elastic matrix and elastic, aligned fibers. In the formulation, small elastic strains are assumed. An overall damage variable is introduced to model damage in the composite system while two local damage variables are used to model damage in the matrix and fibers. The overall and local damage variables are then related through the matrix and fiber volume fractions. The concept of effective stress and the hypothesis of elastic energy equivalence are used to derive the transformation equations between the damaged configuration and the fictitious undamaged configuration. In addition, new expressions are derived for the stress and strain concentration factors of the damaged composite.

The model is applied to a unidirectional fiber-reinforced thin lamina subjected to uniaxial tension. It is also applied to a unidirectional thin lamina subjected to a state of plane stress. In these examples, explicit expressions are derived for the overall and local damage effect tensors through their matrix representations. The relations between the damaged and undamaged elastic properties are also derived using the proposed model. Stress and strain concentration factors for the damaged matrix and fibers are also derived in terms of the undamaged concentration factors and the damage variables. The research presented in this work is the three-dimensional generalization of the uniaxial tension model derived previously by the authors (Kattan and Voyiadjis, 1993).

In this manuscript, it is the aim of the authors to provide at least one example where an analytical solution is possible. The authors are currently engaged in the finite element implementation of this theory in order to tackle more complicated problems and simulate actual damage processes. In addition, the authors have conducted experiments on titanium plates reinforced with SiC fibers in order to validate the numerical results. The authors wish to limit the present manuscript to the theoretical formulation of the theory and present the results of their numerical and experimental work in a forthcoming paper (for more details see Voyiadjis *et al.*, 1993).

Acknowledgements—The research described in this paper was sponsored by the Air Force Office of Scientific Research under Grants AFOSR-90-0227 and F49620-92-J-0463. The authors wish to acknowledge the support and encouragement of Dr Walter F. Jones.

REFERENCES

Allen, D. H. and Harris, C. E. (1987). A thermomechanical constitutive theory for elastic composites with distributed damage—I. Theoretical formulation. *Int. J. Solids Structures* **23**, 1301–1318.

Allen, D. H., Harris, C. E. and Groves, S. E. (1987). A thermomechanical constitutive theory for elastic composites with distributed damage—II. Application to matrix cracking in laminated composites. *Int. J. Solids Structures* **23**, 1319–1338.

Allen, D. H., Harris, C. E., Groves, S. E. and Norvel, R. G. (1988). Characteristics of stiffness loss in crossply laminates with curved matrix cracks. *J. Compos. Mater.* **22**, 71–80.

Bahei-El-Din, Y. A. and Dvorak, G. J. (1989). A reivew of plasticity theory of fibrous composite materials. In *Metal Matrix Composites: Testing, Analysis and Failure Modes* (Edited by W. S. Johnson), pp. 103–129. ASTM STP 1032, Philadelphia, Pennsylvania.

Chaboche, J. L. (1988a). Continuum damage mechanics. Part I: General concepts. *J. Appl. Mech.* **55**, 59–64.

Chaboche, J. L. (1988b). Continuum damage mechanics. Part II: Damage growth, crack initiation and crack growth. *J. Appl. Mech.* **55**, 65–72.

Christensen, R. M. (1988). Tensor transformations and failure criteria for the analysis of fiber composite materials. *J. Compos. Mater.* **22**, 874–897.

Christensen, R. M. (1990). Tensor transformations and failure criteria for the analysis of fiber composite materials. Part II: Necessary and sufficient conditions for laminate failure. *J. Compos. Mater.* **24**, 796–800.

Dvorak, G. J. and Bahei-El-Din, Y. A. (1979). Elastic–plastic behavior of fibrous composites. *J. Mech. Phys. Solids* **27**, 51–72.

Dvorak, G. J. and Bahei-El-Din, Y. A. (1982). Plasticity analysis of fibrous composites. *J. Appl. Mech.* **49**, 327–335.

Dvorak, G. J. and Bahei-El-Din, Y. A. (1987). A bimodal plasticity theory of fibrous composite materials. *Acta Mech.* **69**, 219–241.

Dvorak, G. J. and Laws, N. (1987). Analysis of progressive matrix cracking in composite laminates—II. First ply failure. *J. Compos. Mater.* **21**, 309–329.

Dvorak, G. J., Laws, N. and Hejazi, M. (1985). Analysis of progressive matrix cracking in composite laminates—I. Thermoelastic properties of a ply with cracks. *J. Compos. Mater.* **19**, 216–234.

Hill, R. (1965). A self-consistent mechanics of composite material. *J. Mech. Phys. Solids* **13**, 213–222.

Hill, R. (1972). On constitutive macro-variables for heterogeneous solids at finite strain. *Proc. Roy. Soc. Lond.* **A326**, 131–147.

Kachanov, L. M. (1958). On the creep fracture time. *Izv Akad. Nauk USSR Otd Tekh.* **8**, 26–31 (in Russian).

Kattan, P. I. and Voyiadjis, G. Z. (1990). A coupled theory of damage mechanics and finite strain elasto-plasticity. Part I: Damage and elastic deformations. *Int. J. Engng Sci.* **28**(5), 421–435.

Kattan, P. I. and Voyiadjis, G. Z. (1993). Micromechanical analysis of damaged in uniaxially loaded unidirectional fiber-reinforced laminae. *Int. J. Solids Structures* **30**(1), 16–36.

Krajcinovic, D. (1983). Constitutive equations for damaging materials. *J. Appl. Mech.* **50**, 355–360.

Krajcinovic, D. (1984). Continuum damage mechanics. *Appl. Mech. Rev.* **37**(1), 1–6.

Laws, N. and Dvorak, G. J. (1987). The effect of fiber breaks and aligned penny-shaped cracks on the stiffness and energy release rates in unidirectional composites. *Int. J. Solids Structures* **23**(9), 1269–1283.

Lee, J. W., Allen, D. H. and Harris, C. E. (1989). Internal state variable approach for predicting stiffness reduction in fibrous laminated composites with matrix cracks. *J. Compos. Mater.* **23**, 1273–1291.

Lee, H., Peng, K. and Wang, J. (1985). An anisotropic damage criterion for deformation instability and its application to forming limit analysis of metal plates. *Engng Fract. Mech.* **21**, 1031–1054.

Lemaitre, J. (1985). A continuous damage mechanics model for ductile fracture. *J. Engng Mater. Technol.* **107**, 83–89.

Lemaitre, J. (1986). Local approach of fracture. *Engng Fract. Mech.* **25**(5/6), 523–537.

Lene, F. (1986). Damage constitutive relations for composite materials. *Engng Fract. Mech.* **25**(5/6), 713–728.

Mori, T. and Tanaka, K. (1973). Average stress in matrix and average elastic energy of materials with misfitting inclusions. *Acta Met.* **21**, 571–574.

Murakami, S. (1988). Mechanical modeling of material damage. *J. Appl. Mech.* **55**, 280–286.

Shen, W., Raio, B. and Lee, H. (1985). A crack-damage mechanics model for composite laminates. *Engng Fract. Mech.* **21**(5), 1019–1029.

Sidoroff, F. (1981). Description of anisotropic damage application to elasticity. In *IUTAM Colloquium on Physical Nonlinearities in Structural Analysis* (Edited by J. Hult and J. Lemaitre), pp. 237–244. Springer, Berlin.

Talreja, R. (1985). A continuum mechanics characterization of damage in composite materials. *Proc. Roy. Soc., Lond.* **A399**, 195–216.

Talreja, R. (1986). Stiffness properties of composite laminates with matrix cracking and interior delamination. *Engng Fract. Mech.* **25**(5/6), 751–764.

Voyiadjis, G. Z. and Kattan, P. I. (1992). A plasticity-damage theory for large deformation of solids. Part I: Theoretical formulation. *Int. J. Engng Sci.* **30**(9), 1089–1108.

Voyiadjis, G. Z., Kattan, P. I., Venson, A. R. and Park, T. (1993). Anisotropic Damage Mechanics Modeling in Metal Matrix Composites, Final Report submitted to the Air Force Office of Scientific Research, 141 pages.

APPENDIX

$$l_{11} = M_{11}[(M^{R}_{22})^2 + M^{R}_{11}M^{R}_{22} - (M^{R}_{13})^2], \tag{A1}$$

$$l_{12} = 0, \tag{A2}$$

$$l_{13} = -M_{13}M^{R}_{22}M^{R}_{13}, \tag{A3}$$

$$l_{14} = M_{11}(M^{R}_{13})^2, \tag{A4}$$

$$l_{15} = M_{13}[(M^{R}_{22})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2]-2M_{11}M^{R}_{22}M^{R}_{13}, \tag{A5}$$

$$l_{16} = M_{13}(M^{R}_{13})^2, \tag{A6}$$

$$l_{22} = M_{22}[(M^{R}_{11})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2], \tag{A7}$$

$$l_{23} = -M_{13}M^{R}_{11}M^{R}_{13}, \tag{A8}$$

$$l_{24} = M_{22}(M^{R}_{13})^2, \tag{A9}$$

$$l_{25} = M_{13}(M^{R}_{13})^2, \tag{A10}$$

$$l_{26} = M_{13}[(M^{R}_{11})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2]-2M_{22}M^{R}_{11}M^{R}_{13}, \tag{A11}$$

$$l_{33} = M^{R}_{11}M^{R}_{22}(M_{11}+M_{22}), \tag{A12}$$

$$l_{34} = -M_{13}M^{R}_{13}(M^{R}_{11}+M^{R}_{22}), \tag{A13}$$

$$l_{35} = -M^{R}_{22}(M_{11}M^{R}_{13}-2M^{R}_{11}M_{13}+M_{22}M^{R}_{13}), \tag{A14}$$

$$l_{36} = -M^{R}_{11}(M_{22}M^{R}_{13}-2M^{R}_{22}M_{13}+M_{11}M^{R}_{13}), \tag{A15}$$

$$l_{41} = 0, \tag{A16}$$

$$l_{42} = M_{22}(M^{R}_{13})^2, \tag{A17}$$

$$l_{43} = -M_{13}M^{R}_{22}M^{R}_{13}, \tag{A18}$$

$$l_{44} = M_{22}[(M^{R}_{22})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2], \tag{A19}$$

$$l_{45} = M_{13}[(M^{R}_{22})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2], \tag{A20}$$

$$l_{46} = M^{R}_{13}(M_{13}M^{R}_{13}-2M_{22}M^{R}_{22}), \tag{A21}$$

$$l_{51} = M_{13}[(M^{R}_{22})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2], \tag{A22}$$

$$l_{52} = M_{13}(M^{R}_{13}), \tag{A23}$$

$$l_{53} = -M^{R}_{22}M^{R}_{13}(M_{11}+M_{22}), \tag{A24}$$

$$l_{54} = M_{13}M^{R}_{22}(M^{R}_{11}+M^{R}_{22}), \tag{A25}$$

$$l_{55} = M_{22}[(M^{R}_{22})^2+M^{R}_{11}M^{R}_{22}-(M^{R}_{13})^2]+M^{R}_{22}(M_{11}M^{R}_{22}+M_{11}M^{R}_{11}-2M_{13}M^{R}_{13})-M_{11}(M^{R}_{13})^2, \tag{A26}$$

$$l_{56} = M^{R}_{13}(M_{11}M^{R}_{13}+M_{22}M^{R}_{13}-2M_{13}M^{R}_{22}), \tag{A27}$$

$$l_{61} = M_{13}(M^{R}_{13})^2, \tag{A28}$$

$$l_{62} = M_{13}[M^{R}_{11}M^{R}_{22}+(M^{R}_{11})^2-(M^{R}_{13})^2], \tag{A29}$$

$$l_{63} = -M^{R}_{11}M^{R}_{13}(M_{11}+M_{22}), \tag{A30}$$

$$l_{64} = M_{13}M^{R}_{11}(M^{R}_{11}+M^{R}_{22}), \tag{A31}$$

$$l_{65} = M^{R}_{13}(M_{11}M^{R}_{13}+M_{22}M^{R}_{13}-2M^{R}_{11}M_{13}), \tag{A32}$$

$$l_{66} = M_{22}[M^{R}_{11}M^{R}_{22}+(M^{R}_{11})^2-(M^{R}_{13})^2]+M^{R}_{11}(M_{11}M^{R}_{22}+M_{11}M^{R}_{11}-2M_{13}M^{R}_{13})-M_{11}(M^{R}_{13})^2. \tag{A33}$$